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# VARIATIONAL PRINCIPLES ON STATIC-DYNAMIC ANALYSIS OF VISCOELASTIC THIN PLATES WITH APPLICATIONS*

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**Abstract**—In this paper, according to the integral-type constitutive relation of linear viscoelastic materials, the initial-boundary-value problem on the static-dynamic analysis of viscoelastic thin plates is established by introducing a "structural function". The corresponding variational principles are presented by means of convolution bilinear forms. As applications, we consider the quasi-static responses of a simply-supported square plate with three different load histories in which the classical Ritz method on the spatial response and the interpolation technique of Legendre polynomials on the temporal response are used. The obtained results are compared with the analytical solutions given in this paper. One can see that the approximate solutions agree well with the analytical solutions. © 1998 Elsevier Science Ltd. All rights reserved.

## 1. INTRODUCTION

The inverse variational problem in the calculus of variations is one of the important problems that has been of interest to a great many mechanics workers (Dai Tian-min, 1995). In the theory of viscoelasticity, Gurtin (1963) transformed the initial-boundary-value problem into an equivalent boundary-value problem with the help of the idea of a convolution product and constructed the ruling operator of the boundary-value problem to make it become a symmetrical operator for the selected bilinear form. Hence, the corresponding functional was given. Reddy (1976) directly constructed a simplified Gurtin's type functional for viscoelastic dynamic problems by using a convolution bilinear form. Luo En (1990) further generalized the simplified Gurtin's type variational principle. Dall'Asta and Menditto (1994) studied the inverse variational problem on a perturbed viscoelastic body. However, this is rare for variational principles of special viscoelastic structures. The main cause is that it is impossible that the general variation principles of 3-D-viscoelastic body are simply degenerated to obtain those of special structures. Usually, the special feature of structures will make the ruling operator of a problem become more complex or/and nonsymmetrical and the operator is essentially different from that of a 3-D-viscoelastic body. Hence, this greatly increases the difficulty in constructing the corresponding functional. Dall'Asta and Menditto (1993) pointed this out when they studied the variational problem of a perturbed viscoelastic body. This paper is devoted to the study of the inverse variational problem for the static and dynamic analysis of viscoelastic thin plates in order to provide an available analysis and avoid the additional errors due to the numerical transform method. It is also possible to overcome the limitation of the transform method. To this end, from the Boltzmann relaxation law of the 3-D-linear theory of viscoelasticity, the initial-boundary-value problem for the static-dynamic analysis of thin plates is established. In the deriving process, we use the Laplace transform and its inverse transform and introduce a structural function that depends on the relaxation functions of the given material and expresses the structure feature of a viscoelastic thin plate. Therefore, the constitutive relations of viscoelastic thin plates are obtained from the 3-D-integral type

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constitutive relation. This is the key to establishing the initial-boundary-value problem of viscoelastic thin plates. Then, with the help of the Boltzmann operator (Leitman and Fisher, 1973) and convolution bilinear forms as well as the structural function introduced, the operator ruling the problem may be symmetrized. Hence, the convolution type functionals can be obtained from two different ways. One can see that the forms of the functionals are simpler and more convenient for computation. Until now, variational methods have rarely been applied to the numerical computations in viscoelastic problems. Hence, as the application of the simplified Gurtin's type variation principles in this paper, the quasi-static responses of a simply-supported square plate with three different load histories are analyzed by using the classical Ritz method and the interpolation technique of Legendre polynomials. The three loads are: (i) a step load; (ii) an exponential load; and (iii) an alternating load, respectively. For loads (i) and (ii), we numerically solve the deflections of the plate at different times and compare them with the analytical solutions obtained in this paper. One can see that the numerical results agree well with the analytical ones. For load (iii), it is impossible to apply the transform method to analyze the problem. Hence, the principles and methods proposed in this paper may be widely applied to the static-dynamical analysis of viscoelastic thin plates.

## 2. INITIAL-BOUNDARY-VALUE PROBLEM OF VISCOELASTIC PLATES

Consider a viscoelastic thin plate with the thickness $h$. Assume that the coordinate plane $ox_1x_2$ coincides with the mid-plane undeformed and the $ox_3$-axis is perpendicular to the mid-plane. Hence, the undeformed plate occupies the region to be $B = \{(x_\alpha; x_3) : x_\alpha \in \Omega, |x_3| \leqslant (h/2)\}$ and its edge is $\partial\Omega = \partial\Omega_u + \partial\Omega_\sigma$, in which, $\partial\Omega_u$ and $\partial\Omega_\sigma$ are the portions of the edge given edge displacements and given edge forces, respectively. Letting the displacements at any point in the midplane be $u_i(x_\alpha, t)$ and the stress $\sigma_{ij}(x_k, t)$ and strain $\varepsilon_{ij}(x_k, t)$ (hereafter, the Greek subscript has the ranges 1 and 2 and the Latin subscript the ranges 1, 2 and 3), then we have the equations and conditions as follows.

### 2.1. *Constitutive equations*

For an isotropic linear viscoelastic material, the Boltzmann relaxation law is given as (Leitman and Fisher, 1973)

$$S_{ij} = G_1 \otimes e_{ij}, \quad \sigma_{kk} = G_2 \otimes \varepsilon_{kk} \tag{1}$$

in which, $S_{ij}$ and $e_{ij}$ are the deviatoric tensors of stress and strain, $G_1$ and $G_2$ are the relaxation functions of material and the symbol $\otimes$ expresses the linear Boltzmann operator defined by (Leitman and Fisher, 1973)

$$g(t) \otimes u(t) = g(0)u(t) + \dot{g}(t) * u(t) = g(0)u(t) + \int_0^t \dot{g}(t-\tau)u(\tau)\,\mathrm{d}\tau \tag{2}$$

and the symbol $*$ is the convolution product, $(\dot{\cdot}) = \mathrm{d}(\cdot)/\mathrm{d}\tau$. For convenience, we shall omit the spatial variables in all expressions.

According to the classical theory of plates, the effect of $\sigma_{33}$ on the deformation may be neglected and hence, we obtain from $\sigma_{33} \approx 0$ and (1)

$$(2G_1 + G_2) \otimes \varepsilon_{33} = -(G_2 - G_1) \otimes \varepsilon_{\gamma\gamma} \tag{3}$$

Operating the Laplace transform and its inverse transform on (3), this yields

$$\varepsilon_{33}(t) = -f(t) \otimes \varepsilon_{\gamma\gamma}(t) \tag{4a}$$

in which,

$$f(t) = L^{-1}[(\bar{G}_2-\bar{G}_1)/s(2\bar{G}_1+\bar{G}_2)] \tag{4b}$$

$\bar{u}$ is the Laplace transform of the function $u$ and $L^{-1}$ the Laplace inverse transform, $s$ the transform parameter. Using (1), (3) and (4) as well as the law of composition of the Boltzmann operator (Christensen, 1982), it is not difficult to obtain

$$\sigma_{\alpha\beta} = G_1 \otimes \varepsilon_{\alpha\beta}+\delta_{\alpha\beta}G_3 \otimes \varepsilon_{\gamma\gamma} \tag{5}$$

where, $G_3(t)$ is defined as

$$G_3(t) = G_1(t) \otimes f(t) \tag{6}$$

From (6), we see that the function $G_3(t)$ only depends on the relaxation functions $G_1(t)$ and $G_2(t)$, it itself is not an independent material function, and one can also see that the function $G_3$ plays a key role in establishing the initial-boundary-value problem and the corresponding Gurtin's type variational principle of viscoelastic plates. As $G_3$ expresses the structural feature of a viscoelastic plate, we call it a "structural function".

### 2.2. *Geometry and motion equations*

In the linear theory of plates, we have

$$\varepsilon_{\alpha\beta} = -u_{3,\alpha\beta}x_3 \tag{7}$$

substituting (7) into (5), this yields

$$\sigma_{\alpha\beta} = -x_3(G_1 \otimes u_{3,\alpha\beta}+\delta_{\alpha\beta}G_3 \otimes u_{3,\gamma\gamma}) \tag{8}$$

If the effect of the rotation and inertia in the midplane may be neglected, the motion equation of the plate is given as

$$M_{\alpha\beta,\alpha\beta}+q = \rho h\ddot{u}_3 \tag{9}$$

in which, the internal force moment $M_{\alpha\beta}$ is defined by

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}x_3\,\mathrm{d}x_3 = -\frac{h^3}{12}(G_1 \otimes u_{3,\alpha\beta}+\delta_{\alpha\beta}G_3 \otimes u_{3,\gamma\gamma}) \tag{10}$$

where we have used the relation (8). Substituting the above expression into (9), this yields

$$\frac{h^3}{12}(G_1+G_3) \otimes \nabla^4 u_3+\rho h\ddot{u}_3 = q \quad \text{in } \Omega\times(0,T) \tag{11}$$

### 2.3. *Boundary and initial conditions*

Assume that on $\partial\Omega_u$ the displacements are given and that on $\partial\Omega_\sigma$ the forces are given, then we have the following boundary conditions

$$u_3 = \tilde{u}_3, \quad u_{3,n} = \tilde{\theta} \quad \text{on } \partial\Omega_u\times[0,T] \tag{12a}$$

$$M_n = \tilde{M}_n, \quad V_n = \tilde{V}_n \quad \text{on } \partial\Omega_\sigma\times[0,T] \tag{12b}$$

in which, $\tilde{u}_3$ and $\tilde{\theta}$ are the known deflection and rotation angle on $\partial\Omega_u$ and $\tilde{M}_n$ and $\tilde{V}_n$ the known bending moment and shear force on $\partial\Omega_\sigma$, and also $M_n$ and $V_n$ are given as

$$M_n = -\frac{h^3}{12}(G_1 \otimes u_{3,nn} - G_3 \otimes \nabla^2 u_3)$$

$$V_n = -\frac{h^3}{12}[(G_1+G_3)\otimes(\nabla^2 u_3)_{,n} + G_1 \otimes (u_{,3ns} - u_{3,s}/\rho_s)_{,s}] \tag{13}$$

where $\rho_s$ is the radius of curvature of the edge, $s$ the arc length. In fact, from the definitions of bending moments [see (10)] and shear forces and the constitutive eqns (8), it is not difficult to obtain the expressions (13) by the method similar to deriving the corresponding ones of elastic thin plates (Chien Wei-zhang, 1980). Assuming that the material and structure are in natural states when $t \in (-\infty, 0^-]$ and letting $u_3^0$ and $\dot{u}_3^0$ be the values of $u_3$ and $\dot{u}_3 = \partial u_3/\partial t$ at the initial time $t = 0$, then the initial conditions are

$$u_3 = \dot{u}_3 = 0 \quad \text{in } \bar{\Omega}\times(-\infty, 0^-] \quad u_3|_{t=0} = u_3^0, \quad \dot{u}_3|_{t=0} = \dot{u}_3^0 \quad \text{in } \bar{\Omega}\times t = 0 \tag{14}$$

in which, both the functions $u_3^0$ and $\dot{u}_3^0$ are the only known functions in $x_\alpha$.

Thus, eqn (11) and conditions (12) and (14) form the initial-boundary-value problem for the static-dynamic analysis of viscoelastic thin plates.

## 3. INVERSE VARIATIONAL PROBLEM

Now, the displacement $u_3$ and the coordinates $x_1$, $x_2$, $x_3$, are denoted by $w$, $x$, $y$, $z$, respectively. The variational principle holds as follows.

Variational principle I: The solution of the problem (11), (12), (14) is equivalent to seeking the stationary point of the functional $\Pi_1$ among all $w$ satisfying (12a), and $\Pi_1$ is given as

$$\Pi_1 = \Pi_{1w} + \Pi_{1b} + \Pi_{1t}$$

$$\Pi_{1w} = \frac{h^3}{24}\iint_\Omega [(G_1+G_3)\otimes(w_{,xx}+w_{,yy}) * (w_{,xx}+w_{,yy}) + 2G_1 \otimes (w_{,xy} * w_{,xy} - w_{,xx} * w_{,yy})]\,dx\,dy$$

$$\Pi_{1b} = -\iint_\Omega q * w\,dx\,dy + \int_{\partial\Omega_\sigma} (\tilde{M}_n * w_{,n} - \tilde{V}_n * w)\,ds,$$

$$\Pi_{1t} = \iint_\Omega \rho h\left[\frac{1}{2}\dot{w} * \dot{w} + (w|_{t=0} - w^0)\dot{w}|_{t=T} - \dot{w}^0 w|_{t=T}\right]dx\,dy \tag{15}$$

where $w^0$ and $\dot{w}^0$ are the initial values of the functions $w$ and $\dot{w}$.

*Proof*: Observing the Boltzmann operator has the property

$$(A \otimes B) * C = A \otimes B * C = A \otimes (B * C) = A \otimes C * B \tag{16}$$

it can be obtained

$$\delta\Pi_{1w}^{(1)} = \delta\left\{\frac{h^3}{24}\iint_\Omega [(G_1+G_3)\otimes(w_{,xx} * w_{,xx} + w_{,yy} * w_{,yy} + 2w_{,xy} * w_{,xy})]\,dx\,dy\right\}$$

$$= \frac{h^3}{12}\iint_\Omega [(G_1+G_3)\otimes(w_{,xx} * \delta w_{,xx} + w_{,yy} * \delta w_{,yy} + 2w_{,xy} * \delta w_{,xy})]\,dx\,dy$$

$$= \frac{h^3}{12}\iint_{\Omega}(G_1+G_3)\otimes(\nabla^4 w)*\delta w\,\mathrm{d}x\,\mathrm{d}y$$

$$+\frac{h^3}{12}\int_{\partial\Omega}(G_1+G_3)\otimes\left\{w_{,nn}*\delta w_{,n}-\left[(\nabla^2 w+w_{,ss})_{,n}\right.\right.$$

$$\left.\left.-\left(\frac{1}{\rho_s}w_{,s}\right)_{,s}\right]*\delta w\right\}\mathrm{d}s-\frac{h^3}{12}(G_1+G_3)\otimes\left[\sum_{k=1}^{i}\Delta\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)\right]_k*\delta w_k$$

$$\delta\Pi_{1w}^{(2)}=\delta\left\{\frac{h^3}{24}\iint_{\Omega}(G_1+G_3)\otimes(w_{,xx}+w_{,yy})*(w_{,xx}+w_{,yy})\,\mathrm{d}x\,\mathrm{d}y\right\}$$

$$=\frac{h^3}{12}\iint_{\Omega}(G_1+G_3)\otimes\nabla^2 w*(\delta w_{,xx}+\delta w_{,yy})\,\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{h^3}{12}\iint_{\Omega}(G_1+G_3)\otimes\nabla^4 w*\delta w\,\mathrm{d}x\,\mathrm{d}y$$

$$+\frac{h^3}{12}\int_{\partial\Omega}(G_1+G_3)\otimes[\nabla^2 w*\delta w_{,n}-(\nabla^2 w)_{,n}*\delta w]\,\mathrm{d}s$$

$$\delta\Pi_{1w}^{(3)}=\delta\left\{\frac{h^3}{24}\iint_{\Omega}2(G_1+G_3)\otimes(w_{,xy}*w_{,xy}-w_{,xx}*w_{,yy})\,\mathrm{d}x\,\mathrm{d}y\right\}$$

$$=\delta(\Pi_{1w}^{(1)}-\Pi_{1w}^{(2)})=\delta\Pi_{1w}^{(1)}-\delta\Pi_{1w}^{(2)}$$

$$=\frac{h^3}{12}\int_{\partial\Omega}(G_1+G_3)\otimes\left\{(w_{,nn}-\nabla^2 w)*\delta w_{,n}-\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_{,s}*\delta w\right\}\mathrm{d}s$$

$$-\frac{h^3}{12}(G_1+G_3)\otimes\left[\sum_{k=1}^{i}\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_k*\delta w_k\right]$$

$$\delta\Pi_{1w}^{(4)}=\delta\left\{\frac{h^3}{24}\iint_{\Omega}2G_1\otimes(w_{,xy}*w_{,xy}-w_{,xx}*w_{,yy})\,\mathrm{d}x\,\mathrm{d}y\right\}$$

$$=\frac{h^3}{12}\int_{\partial\Omega}G_1\otimes\left[(w_{,nn}-\nabla^2 w)*\delta w_{,n}-\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_{,s}*\delta w\right]\mathrm{d}s$$

$$-\frac{h^3}{12}G_1\otimes\left[\sum_{k=1}^{i}\Delta\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_k*\delta w_k\right]$$

Hence, it is easy to see that

$$\delta\Pi_{1w}=\delta\Pi_{1w}^{(2)}+\delta\Pi_{1w}^{(4)}=\frac{h^3}{12}\iint_{\Omega}(G_1+G_3)\otimes\nabla^4 w*\delta w\,\mathrm{d}x\,\mathrm{d}y$$

$$-\frac{h^3}{12}\int_{\partial\Omega}-[G_3\otimes\nabla^2 w+G_1\otimes w_{,nn}]*\delta w_{,n}\,\mathrm{d}s$$

$$+\frac{h^3}{12}\int_{\partial\Omega}-\left[(G_1+G_3)\otimes(\nabla^2 w)_{,n}+G_1\otimes\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_{,s}\right]*\delta w\,ds$$

$$-\frac{h^3}{12}G_1\otimes\left[\sum_{k=1}^{i}\Delta\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_k*\delta w_k\right]$$

$$=\frac{h^3}{12}\iint_{\Omega}(G_1+G_3)\otimes\nabla^4 w*\delta w\,dx\,dy-\int_{\partial\Omega_\sigma}[M_n*\delta w_{,n}-(Q_n+M_{ns,n})*\delta w]\,ds$$

$$-\frac{h^3}{12}G_1\otimes\left[\sum_{k=1}^{i}\Delta\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_k*\delta w_k\right] \tag{17a}$$

In the expression (17a), $\Delta(\cdot)_k$ is the jumping value of the function $(\cdot)$ passing the corner point $k$ of the edge. In deriving (17a), we have also applied the Green formula, the properties of the convolution product and the Boltzmann operator. In addition, we also have

$$\delta\Pi_{1b}=-\iint_{\Omega}q*\delta w\,dx\,dy+\int_{\partial\Omega_\sigma}(\tilde{M}_n*\delta w_{,n}-\tilde{V}_n*\delta w)\,ds \tag{17b}$$

$$\delta\Pi_{1t}=\iint_{\Omega}\rho h\ddot{w}*\delta w\,dx\,dy+\iint_{\Omega}\rho h(w|_{t=0}-w^0)\delta\dot{w}|_{t=T}\,dx\,dy$$

$$+\iint_{\Omega}\rho h(\dot{w}|_{t=o}-\dot{w}^0)\delta w|_{t=T}\,dx\,dy \tag{17c}$$

In fact, it is not difficult to prove that (Dall'Asta and Menditto, 1994)

$$\delta\left\{\iint_{\Omega}\frac{1}{2}\rho h\dot{w}*\dot{w}\,dx\,dy\right\}=\iint_{\Omega}\rho h\dot{w}*\delta\dot{w}\,dx\,dy$$

$$=\iint_{\Omega}\rho h\ddot{w}*\delta w\,dx\,dy+\iint_{\Omega}\rho h\dot{w}|_{t=0}\delta w|_{t=T}\,dx\,dy-\iint_{\Omega}\rho h\dot{w}|_{t=T}\delta w|_{t=0}\,dx\,dy$$

Substituting (17) into $\delta\Pi_1=\delta\Pi_{1w}+\delta\Pi_{1b}+\delta\Pi_{1t}$ and letting $\delta\Pi_1=0$, we obtain the variational equation

$$\delta\Pi_1=\frac{h^3}{12}\iint_{\Omega}(G_1+G_3)\otimes(\nabla^4 w-q+\rho h\ddot{w})*\delta w\,dx\,dy$$

$$-\int_{\partial\Omega_\sigma}[(M_n-\tilde{M}_n)*\delta w_{,n}+(V_n-\tilde{V}_n)*\delta w]\,ds$$

$$+\iint_{\Omega}\rho h(w|_{t=0}-w^0)\delta\dot{w}|_{t=T}\,dx\,dy+\iint_{\Omega}\rho h(\dot{w}|_{t=0}-\dot{w}^0)\delta w|_{t=T}\,dx\,dy$$

$$-\frac{h^3}{12}G_1*\left[\sum_{k=1}^{i}\Delta\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_k*\delta w_k\right]=0 \tag{18}$$

Observing the arbitrariness of $\delta w$, $(\delta w)_{,n}$, $\delta\dot{w}|_{t=T}$, $\delta w|_{t=T}$ and $\delta w_k$ and using the Titchmarsh theorem (Leitman and Fisher, 1973) and the fundamental preliminary theorem (Chien Wei-zhang, 1980) of the calculus of variations, this yields the eqn (11), the boundary conditions (12b), the initial conditions (14) as well as the condition at the corner point $k$

$$\Delta\left(w_{,ns}-\frac{1}{\rho_s}w_{,s}\right)_k=0,\quad k=1,2,\ldots,i \tag{19}$$

For the smooth edge, the condition (19) vanishes. The variational principle I is essentially a simplified Gurtin's type variational principle, in which, both the classical Cartesian bilinear form and the modern convolution bilinear form are used simultaneously. The functional $\Pi_1$ can be made only when the structure function $G_3$ is introduced. As we introduce $G_3$, it is possible that the operator ruling the problem is symmetrized. Next, we shall construct a Gurtin's type functional of the problem (11), (12), (14) by using the other way. For this method, the initial-boundary-value problem may be translated into an equivalent boundary-value problem and makes the operator ruling the boundary-value problem to be symmetrical for the selected convolution bilinear form.

Variational principle II: Let $g_1(t)=t, g_2(t)=t\dot{w}^0+w^0$, then the solution of the problem (11), (12), (14) is equivalent to seeking the stationary point of the functional $\Pi_2$ among all $w$ satisfying (12a) and $\Pi_2$ is given

$$\Pi_2=\Pi_{2w}+\Pi_{2b}+\Pi_{2t}$$

$$\Pi_{2w}=\frac{h^3}{24}\iint_\Omega g_1*[(G_1+G_3)\otimes(w_{,xx}+w_{,yy})*(w_{,xx}+w_{,yy})$$

$$+2G_1\otimes(w_{,xy}*w_{,xy}-w_{,xx}*w_{,yy})]\,dx\,dy$$

$$\Pi_{2b}=-\iint_\Omega g_1*q*w\,dx\,dy+\int_{\partial\Omega_n}g_1*(\tilde{M}_n*w_{,n}-\tilde{V}_n*w)\,ds$$

$$\Pi_{2t}=\iint_\Omega\rho h\left(\frac{1}{2}g_1*w*w+g_2*w\right)dx\,dy \tag{20}$$

*Proof*: First, using the method similar to deriving (17) and observing that both $g_1(t)$ and $g_2(t)$ are known functions, it is not difficult to obtain the variational equation as follows:

$$\delta\Pi_2=\iint_\Omega\left\{g_1*\left[\frac{h^3}{12}(G_1+G_3)*\nabla^4w-q+\rho hw\right]+\rho hg_2\right\}*\delta w\,dx\,dy$$

$$+\int_{\partial\Omega_\sigma}g_1*(V_n-\tilde{V}_n)*\delta w\,ds-\int_{\partial\Omega_\sigma}g_1*(M_n-\tilde{M}_n)*\delta w_{,n}\,ds=0 \tag{21}$$

Observing the arbitrariness of $\delta w$ and $(\delta w)_{,n}$, and using the Titchmarsh theorem, we may obtain the following boundary-value problem

$$g_1*\left[\frac{h^3}{12}(G_1+G_3)\otimes\nabla^4w-q+\rho hw\right]+\rho hg_2=0\quad\text{in }\Omega\times(0,T) \tag{22a}$$

$$g_1*(M_n-\tilde{M}_n)=0,\quad g_1*(V_n-\tilde{V}_n)=0\quad\text{on }\partial\Omega_\sigma\times[0,T] \tag{22b}$$

Second, we have also to prove that the boundary-value problem (22) is equivalent to the initial-boundary-value problem (11), (12), (14). Since the conditions (12a) on $\partial\Omega_u$ have been satisfied previously and using the Titchmarsh theorem for (22b), this yields $M_n = \tilde{M}_n$ and $V = \tilde{V}_n$. Hence, the boundary conditions for the problem (11), (12), (14) identify with those of the problem (22). Now, it is only necessary to prove that eqn (22a) is equivalent to eqn (11) and the condition (14). In fact, operating the Laplace transform on (22a), we obtain

$$\bar{g}_1 \left[ \frac{h^3}{12}(\bar{G}_1 + \bar{G}_2) s \nabla^4 \bar{w} - \bar{q} + \rho h \bar{w} \right] + \rho h \bar{g}_2 = 0$$

Observing the Laplace transform has the property

$$L[f^{(n)}(t)] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0) \qquad (23)$$

then we have the equation

$$\frac{1}{s^2} \left[ \frac{h^3}{12}(\bar{G}_1 + \bar{G}_3) s \nabla^4 w - \bar{q} + \rho h \bar{w} \right] + \rho h \left[ -\frac{1}{s^2} \dot{w}^0 - \frac{1}{s} w^0 \right] = 0$$

that is,

$$\frac{h^3}{12}(\bar{G}_1 + \bar{G}_3) s \nabla^4 w - \bar{q} + \rho h \bar{w} - \rho h (s w^0 + \dot{w}^0) = 0 \qquad (24)$$

One can see that eqn (24) involves the initial conditions (14). The inverse transform of (24) gives the governing eqn (11), and vice versa. Hence, the problem (11), (12), (14) is equivalent to the problem (22). Only the problem (22) implies the initial conditions (14). So far, we have constructed the convolution type functionals $\Pi_1$ and $\Pi_2$ of the problem (11), (12), (14) from two different ways. In the numerical computation, the variational principle II is more complex because the Gurtin's type functional involves a threefold convolution product but the simplified Gurtin's type functional only includes a twofold convolution product and hence, both the error and solving difficulty will be greatly increased provided the variational principle II is applied.

To the authors' knowledge, this is rare for approximate methods on the basis of the convolution type functional. On the one hand, it is not very easy to construct the functional of a class of special viscoelastic structures even if deflections of plates are small. On the other hand, it is impossible to expect to give a general and valid computation method because the viscoelastic problems involve time terms. Hence, it is necessary (Dall'Asta and Menditto, 1994) to provide some special computation methods for a class of problems. Even so this is also very meaningful for engineering and technology. To this end, we try here to propose a method of analyzing the quasi-static bending problem of viscoelastic plates. For this class of problems it is possible to obtain the solution we expect by using the classical variational method. For convenience, letting the Poisson ratio $\nu(t) =$ const, and introducing the dimensionless variables and parameters as follows:

$$\xi = x/R_c \quad \eta = y/R_c, \quad W = w/h, \quad \hat{M}_n = R_c^2 M_n / D(0) h,$$
$$\hat{V}_n = R^3 V_n / D(0) h, \quad \tilde{M}_n^* = R_c^2 \tilde{M}_n / D(0) h, \quad \tilde{V}_n^* = R_c^3 \tilde{V}_n / D(0) h,$$
$$Q = R_c^4 q / D(0) h, \quad e(t) = D(t)/D(0) = E(t)/E(0), \quad D(t) = E(t) h^3 / 12(1-\nu^2) \qquad (25)$$

in which $E(t)$ is a uniaxial relaxation function and $R_c$ the characteristic length of the plate, then we have the boundary-value problem in terms of the dimensionless variables

$$e(t) \otimes (\nabla^4 W) = Q \quad \text{in } \Omega \times (0, T) \qquad (26a)$$

$$W = \tilde{w}/h, \quad W_{,n} = R_c \tilde{\theta}/h \quad \text{on } \partial\Omega_u \times [0, T] \qquad (26b)$$

$$\hat{M}_n = \tilde{M}_n^*, \quad \hat{V}_n = \tilde{V}_n^* \quad \text{on } \partial\Omega_\sigma \times [0, T] \tag{26c}$$

where the dimensionless variables $(\xi, \eta)$ and the region are still denoted by $(x, y)$ and $\Omega$.

Variation principle III: The solution of the problem (26) is equivalent to seeking the stationary point of $\Pi_3$ among all $W$ satisfying (26b) and $\Pi_3$ is given as

$$\Pi_3 = \frac{1}{2}\iint_\Omega e \otimes [(W_{,xx} + W_{,yy}) * (W_{,xx} + W_{,yy}) - 2(1-\nu)(W_{,xx} * W_{,yy} - W_{,xy} * W_{,xy})]\,\mathrm{d}x\,\mathrm{d}y - \iint_\Omega Q * W\,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega_\sigma} (\tilde{M}_n^* * W_{,n} - \tilde{V}_n^* * W)\,\mathrm{d}s \tag{27}$$

For plates with only the simply-supported or/and clamped edge, $\Pi_3$ may be simplified as

$$\Pi_3 = \frac{1}{2}\iint_\Omega e \otimes [(W_{,xx} + W_{,yy}) * (W_{,xx} + W_{,yy})]\,\mathrm{d}x\,\mathrm{d}y - \iint_\Omega Q * W\,\mathrm{d}x\,\mathrm{d}y \tag{27'}$$

In fact, the variational principle III is a simplification of the variational principle I. If the fundamental period of free vibration of the structure is much shorter than the relaxation time of the material, then it is rational to neglect the effect of inertia (Hoff, 1958). In this case the variational principle III may be directly applied to obtain the approximate solution of the problem.

## 4. APPLICATIONS

As applications, we consider the quasi-static responses of a viscoelastic simply-supported square plate with three load histories. Assume that the viscoelastic behaviors of the material may be described by a model with three parameters. Hence, the dimensionless relaxation function $e(t)$ is given as

$$e(t) = A + B\exp(-\alpha t) = \frac{E_2}{E_1 + E_2} + \frac{E_1}{E_1 + E_2}\exp\left(-\frac{E_1 + E_2}{\gamma_2}t\right) \tag{28}$$

in which, $E_1 = E_2 = 3.0 \times 10^4$ MPa, $\gamma_2 = 1.0 \times 10^6$ MPa $\cdot$ days.

Analytical solution of the square plate: under this case, the boundary-value problem is

$$e \otimes \nabla^4 W = Q \quad \text{in } \Omega \times (0, T) \tag{29a}$$

$$W = 0 \quad \text{on } \partial\Omega_u \times [0, T] \tag{29b}$$

$$\hat{M}_n = 0 \quad \text{on } \partial\Omega_\sigma \times [0, T] \tag{29c}$$

Using the Titchmarsh theorem, one can see that the condition (29c) is equivalent to $W_{,xx}|_{x=0,1} = W_{,yy}|_{y=\pm(1/2)} = 0$. Assume that the $Q(x, y, t) = Q_1(x, y) \cdot Q_2(t)$ and that the solution of (29) have the form

$$W(x, y, t) = W_1(x, y) \cdot W_2(t) \tag{30}$$

Substituting (30) into (29), this yields

$$e \otimes W_2 = Q_2 \tag{31}$$

$$\begin{cases} \nabla^4 W_1 = Q_1 & \text{in } \Omega \\ W_1 = W_{,xx} = 0 & \text{on } x = 0, l = a/b \\ W_1 = W_{,yy} = 0 & \text{on } y = \pm l/2 = \pm 1/2 \end{cases} \tag{32}$$

For a uniform load $Q_1(x, y) = 1$, we only consider here the case of $Q_1 = 1$. It is not difficult to obtain the Levy solution of (32)

$$W_1(x, y) = \frac{4l^4}{\pi^5} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^5} \left( 1 - \frac{2 + \alpha_m \operatorname{th} \alpha_m}{2 \operatorname{ch} \alpha_m} \operatorname{ch} \frac{2\alpha_m y}{l} + \frac{\alpha_m}{2 \operatorname{ch} \alpha_m} \left( \frac{2y}{l} \right) \operatorname{sh} \frac{2\alpha_m y}{l} \right) \sin \frac{m\pi x}{l}$$

in which, $\alpha_m = (m\pi/2l)$, for a square plate, $l = a/b = 1$. The maximum value of $W_1(x, y)$ is given as

$$W_{1\max} = W_1(x, y)|_{x=(1/2), y=0} = \frac{4}{\pi^5} \sum_{k=1,3}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5} \left( 1 - \frac{2 + \alpha_m \operatorname{th} \alpha_m}{2 \operatorname{ch} \alpha_m} \right) \tag{33}$$

In order to obtain $W_2(t)$, we have to operate the Laplace transform on (31). As examples, consider the following cases.

1. Let $Q_2(t) = q_0 H(t)$, then $\bar{W}_2(s) = q_0/s^2 \bar{e}(s)$. Observing (28), we obtain

$$\bar{W}_2(s) = \frac{q_0}{(A+B)\left(s + \dfrac{A\alpha}{A+B}\right)} \left(1 + \frac{\alpha}{s}\right)$$

and

$$W_2(t) = L^{-1}[\bar{W}_2(s)] = q_0 \left[ \frac{1}{A} - \frac{B}{A(A+B)} \exp\left( - \frac{A\alpha}{A+B} t \right) \right]$$

If taking $q_0 = 1$, $A = B = 0.5$, $\alpha = 0.06$/days, then

$$W_2(t) = 2 - e^{-0.03t} \tag{34a}$$

2. Let $Q_2(t) = q_0[1 + \exp(-\beta t)]$, then we have

$$\bar{W}_2(s) = q_0 \left( \frac{1}{s} + \frac{1}{s+\beta} \right) \Big/ s\bar{e}(s)$$

$$= \frac{q_0}{A+B} \left\{ \left( 1 - B \exp\left( - \frac{A\alpha}{A+B} s \right) \right) + \frac{1}{(s+\beta)\left(s + \dfrac{A\alpha}{A+B}\right)} (s+\alpha) \right\}$$

and

$$W_2(t) = L^{-1}[\bar{W}_2(s)]$$

$$= \begin{cases} q_0\left(\frac{1}{A} - \frac{B}{A(A+B)}\exp\left(-\frac{A\alpha}{A+B}t\right)\right) + \frac{q_0}{A\alpha-(A+B)\beta} \\ \cdot\left[(\alpha-\beta)\exp(-\beta t) - \frac{B\alpha}{A+B}\exp\left(-\frac{A\alpha}{A+B}t\right)\right] & \text{for } \beta \neq \frac{A\alpha}{(A+B)} \\ q_0\left(\frac{1}{A} - \frac{B}{A(A+B)}\exp\left(-\frac{A\alpha}{A+B}t\right)\right) \\ +\frac{q_0}{A+B}\exp(-\beta t)(1+(\alpha-\beta)t) & \text{for } \beta = \frac{A\alpha}{(A+B)} \end{cases}$$

If taking $A = B = 0.5$, $\alpha = 0.06$/days, $\beta = 0.05$/days, $q_0 = 1$, then

$$W_2(t) = 2+0.5(e^{-0.03t}-e^{-0.05t}) \tag{34b}$$

Approximate solution of the square plate: we shall see that for quasi-static problems of plates, the classical Ritz method may be directly used to obtain the solution we expect. Let the space $U$ of the solutions be

$$U = \{W\in U|\, W = \tilde{w}/h,\, W_{,n} = R_c\tilde{\theta}/h, \text{on } \partial\Omega_u\times[0,T]\} \tag{35}$$

and assuming that $X_i(x)\,Y_j(y)\psi_k(\tau)$ is an orthonormalization system in $U$, then the approximate solution may be expressed as

$$W^{\alpha} = \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=0}^{n} C_{ijk}X_i(x)\,Y_j(y)\psi_k(\tau) \tag{36}$$

in which, $C_{ijk}$ are the undetermined coefficients depending on the integral variable $\tau$ and only depending on the present time T. For custom and visualization, we still denote the present time T by t in the following. Substituting (36) into the expression (27′), and integrating the obtained expression, we have

$$\Pi_3 = \tfrac{1}{2}A_{ijkqrs}C_{ijk}C_{qrs} - B_{qrs}C_{qrs} \tag{37a}$$

where,

$$A_{ijkqrs} = e\otimes(\psi_k*\psi_s)\iint_{\Omega}[(X_i''Y_j+X_iY_j'')(X_q''Y_r+X_qY_r'') \\ -2(1-\nu)(X_i''Y_jX_qY_r''-X_i'Y_j'X_q'Y_r')]\,\mathrm{d}x\,\mathrm{d}y \tag{37b}$$

$$B_{qrs} = \iint_{\Omega}Q*(X_qY_r\psi_s)\,\mathrm{d}x\,\mathrm{d}y - \int_{\partial\Omega_\sigma}[\tilde{M}_n^**(X_qY_r\psi_s)_{,n} - \tilde{V}_n^**(X_qY_r\psi_s)]\,\mathrm{d}s \tag{37c}$$

Let $\delta\Pi_3 = 0$, this yields

$$A_{ijkqrs}C_{ijk} - B_{qrs} = 0, \quad 1\leqslant q\leqslant l, 1\leqslant r\leqslant m, 0\leqslant s\leqslant n \tag{38}$$

This is a system of algebraical equation about $C_{ijk}$ and from it we may obtain the coefficients $C_{ijk}$. For a viscoelastic square plate with the simply-supported edge, the approximate solution $W^{\alpha}$ may be expressed as

$$W^{\alpha}(x, y, \tau) = \Sigma\Sigma\Sigma C_{ijk} \sin(i\pi x) \sin(j\pi y)\psi_k(\tau) \tag{39}$$

where $\psi(\tau)$ is an orthonormalization system of Legendre polynomials in the interval $[0, t]$ given by

$$\psi_k(\tau) = \sqrt{\frac{2k+1}{t}} \frac{t^k}{2^{2k}k!} \frac{d^k}{d\tau^k} \left\{ \left[ \left( \frac{2\tau}{t} - 1 \right)^2 - 1 \right]^k \right\}, \quad k = 0, 1, \ldots, n \tag{40}$$

It is not difficult to prove

$$\psi_k(t) * \psi_s(t) = (-1)^k \delta_{ks} \tag{41}$$

where $\delta_{ks}$ is the Kronecker-$\delta$ symbol and the repeat index $k$ does not denote a summation.

From the above derivation, it is easy to obtain the approximate maximum value of the deflection under the uniform load to be

$$W^{\alpha}_{\max}(t) = W^{\alpha}_1 W^{\alpha}_2(t) \tag{42a}$$

in which,

$$W^{\alpha}_1 = \frac{4}{\pi^b} \sum_{i=1,3}^{l} \sum_{j=1,3}^{m} (-1)^{\frac{i+j}{2}-1} (ij)^{-1} (i^2+j^2)^{-2} (1-\cos(i\pi))(1-\cos(j\pi)) \tag{42b}$$

$$W^{\alpha}_2 = \sum_{k=1}^{n} (-1)^k F_k(t)\psi_k(t) \tag{42c}$$

$$F_k(t) = Q_2 * \psi_k(t) = \sqrt{\frac{2k+1}{t}} \sum_{s=0}^{[k/2]} \frac{(-1)^s (2k-2s)!}{2^k s!(k-s)!(k-2s)!} \int_0^t \left(1 - \frac{2\tau}{t}\right)^{k-2s} Q_2(\tau)\,d\tau \tag{42d}$$

In order to compare the analytical solution (33) with the approximate solution (42), we shall give the numerical results in three load histories.

Application 1: Let $Q(t) = H(t)$, then we have

$$W^{\alpha}_2(t) = 1/e(t), \quad W_2(t) = 2-\exp(-0.03t), \quad R_c = (W^{\alpha}_{\max} - W_{\max})/W_{\max} \tag{43}$$

in which, $W_2(t)$ is given by (34a). The computation results are listed in Table 1.

In Table 1, $R_t = \gamma_2/(E_1+E_2)$ is the relaxation time of the material. In computation, we take two terms and one term for the spatial and temporal variables in the formula (36), that is, $l$, $m = 1, 3$; $n = 0$. We have to point out that $F_k = 0$ in (42) when $k \geqslant 1$. In other words, it is impossible that the degree of accuracy is increased by means of increasing the number of terms of Legendre polynomials for the step load. This is different from Dall'Asta and Menditto (1993). One can also see that, from Table 1, the approximate solutions agree well with the analytical solutions given in this paper.

Application 2: Let $Q(t) = 1+\exp(-\beta t)$, $\beta = 0.05$, we have

Table 1. The maximum values of deflection for step load (i)

| $t/R_t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $W^{\alpha} \times 10^{-3}$ | 4.055 | 5.929 | 7.144 | 7.726 | 7.965 | 8.057 | 8.091 | 8.103 | 8.108 | 8.110 | 8.111 | 8.111 | 8.111 |
| $W \times 10^{-3}$ | 4.060 | 5.657 | 6.626 | 7.214 | 7.571 | 7.787 | 7.918 | 7.997 | 8.046 | 8.075 | 8.093 | 8.103 | 8.110 |
| $R_e$% | −0.11 | 4.81 | 7.81 | 7.10 | 5.21 | 3.47 | 2.18 | 1.32 | 0.78 | 0.43 | 0.22 | 0.09 | 0.01 |

$$F_k(\eta) = \int_0^\eta \psi_k(t-\tau)Q_2(\tau)\,\mathrm{d}\tau$$

$$= \sqrt{\frac{2k+1}{t}} \sum_{s=0}^{[k/2]} \sum_{p=0}^{k-2s} \frac{(-1)^{p+s}(2k-2s)!s_p(\eta)}{2^{k-p}t^p s!(s-k)!(k-p-2s)!p!}$$

$$s_p(\eta) = \int_0^\eta \tau^p Q_2(\tau)\,\mathrm{d}t$$

$$= \begin{cases} \eta + (1-\exp(-\beta\eta))/\beta, & \text{for } p = 0 \\ \frac{1}{\beta^{p+1}}\{p! - \exp(-\beta\eta)[(\beta\eta)^p + n(\beta\eta)^{p-1} + n(n-1)(\beta\eta)^{p-2} + \cdots + n!]\} & \\ + \frac{\eta^{p+1}}{(p+1)} & \text{for } p \geqslant 1 \end{cases} \tag{44a}$$

In addition, $W_2(t)$ is given by (34b), namely

$$W_2(t) = 2 + 0.5[e^{-0.03t} - e^{-0.05t}] \tag{44b}$$

In computation, we take $l, m = 1,3$; $n = 0, 1, 2, 3, 4, 5$ in the formula (36). The numerical results are listed in Table 2, in which, $W_k^\alpha$ is the approximate solution and $W_{\max} = W_5^\alpha$ is the analytical solution. One can see that the variational solutions approach the analytical solution with increasing number of terms.

Application 3: Let

$$Q(t) = \begin{cases} H(t) + \exp(-\beta t) & t \in [0, t_1] \\ H(t_1) + \exp(-\beta t_1) & t \in [t_1, +\infty) \end{cases} \tag{45}$$

where $\beta = 0.05$, $t_1 = N \cdot R_t$, $N$ is a positive integer. For the load history, it is impossible to analyze the quasi-static response of a viscoelastic structure with the help of the transform method. Hence, the relative reports have not been found. Here, we only intend to show some features of the class of complex problems. In Fig. 1, the curves of deflection of the viscoelastic square plate are shown when $N = 2, 3, 4$. It is necessary to point out that if $t \leqslant t_1$ the upper limit $\eta$ of the integrations in (44a) is $t$, namely, $\eta = t$, otherwise, $\eta = t_1$. Figure 1 shows some features of the deformation of the plate. If we unload at the time $t = t_1$, the bigger $N$ is, the lower the ratio of deflection is. One can see that for the larger loading time $t_1$, the springback of the plate is much slower after unloading. The cause of

Table 2. The maximum values of deflection under exponential load (ii)

| $t/R_t$ | $W_0^\alpha \times 10^{-3}$ | $W_1^\alpha \times 10^{-3}$ | $W_2^\alpha \times 10^{-3}$ | $W_3^\alpha \times 10^{-3}$ | $W_4^\alpha \times 10^{-3}$ | $W_5^\alpha \times 10^{-3}$ | $W_{\max} \times 10^{-3}$ | $R_e\%$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 8.111 | 8.111 | 8.111 | 8.111 | 8.111 | 8.111 | 8.120 | −0.11 |
| 1 | 9.953 | 8.295 | 8.524 | 8.505 | 8.506 | 8.507 | 8.469 | +0.44 |
| 2 | 10.062 | 7.849 | 8.604 | 8.480 | 8.494 | 8.493 | 8.483 | +0.12 |
| 3 | 10.563 | 7.339 | 8.626 | 8.312 | 8.367 | 8.359 | 8.406 | −0.56 |
| 4 | 10.027 | 6.993 | 8.684 | 8.144 | 8.269 | 8.246 | 8.322 | −0.91 |
| 5 | 9.960 | 6.811 | 8.770 | 8.004 | 8.223 | 8.174 | 8.255 | −0.99 |
| 6 | 9.698 | 6.740 | 8.859 | 7.890 | 8.216 | 8.130 | 8.207 | −0.95 |
| 7 | 9.489 | 6.733 | 8.936 | 7.794 | 8.234 | 8.099 | 8.175 | −0.94 |
| 8 | 9.323 | 6.763 | 8.996 | 7.712 | 8.265 | 8.075 | 8.155 | −0.98 |
| 9 | 9.191 | 6.809 | 9.038 | 7.642 | 8.304 | 8.051 | 8.141 | −1.11 |
| 10 | 9.083 | 6.864 | 9.065 | 7.583 | 8.345 | 8.028 | 8.133 | −1.29 |
| 11 | 8.995 | 6.920 | 9.079 | 7.534 | 8.368 | 8.003 | 8.128 | −1.54 |
| 12 | 8.922 | 6.975 | 9.084 | 7.493 | 8.426 | 7.977 | 8.125 | −1.81 |

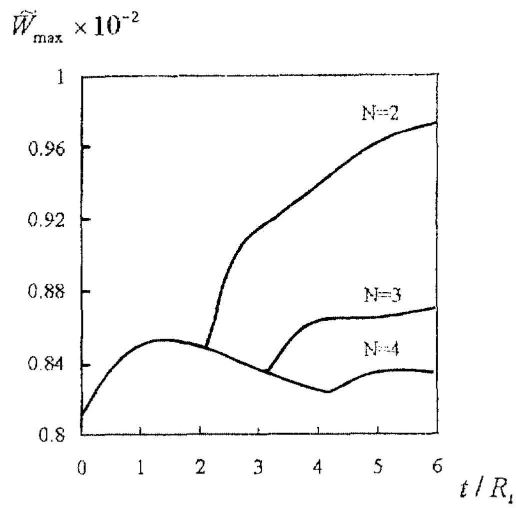


Fig. 1. The curves of deflection of square plate for three loads.

this is that the dissipation of energy of the plate is more and more and the strain energy stored is less and less with the increase of loading time.

## 5. CONCLUSIONS

From all the analyses above, we come to the conclusions as follows:

(1) The initial-boundary-value problem (11), (12), (14) for the static-dynamical analyses of viscoelastic thin plates is established by means of the Boltzmann relaxation law of 3-D-linear theory of viscoelasticity and the introduction of the structural function $G_3$. The initial-boundary-value problem may be applied to the static-dynamical analysis of viscoelastic thin plates with any shape and any load history.

(2) The three variational principles are presented from the Boltzmann operator and convolution bilinear form. Here the structural function $G_3$ plays a key role.

(3) On the basis of the variational principle III, we propose an available method to analyze the quasi-static responses of a viscoelastic simply-supported rectangular or square plate. For comparison, the analytical solution of the problem is also given.

(4) As application, we consider the deflections of the square plate with three load histories and compare the obtained approximate solutions [for loads (i) and (ii)] with the analytical solutions. One can see that they agree well with each other. For load (iii), it is impossible to apply the transform method to analyze the problem. Hence, the method presented in this paper may be widely applied to the static-dynamical analyses of viscoelastic thin plates with any load history.

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